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Disorder in non-equilibrium models

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Abstract

An overview is presented of non-equilibrium systems in the presence of disorder, which typically strongly affects their behaviour. The examples considered here are those for which most progress is currently being made, namely low-dimensional collective models of the lattice-based exclusion type with substitutional disorder. Discussions are given of such effects as localization, and of disorder-induced crossovers to new critical or singular behaviour at steady state non-equilibrium transitions, and to new asymptotic or critical dynamics. Mappings to disordered quantum spin systems are used to provide some exact results (exploiting Goldstone symmetries or free fermion equivalences) and to introduce non-equilibrium/disorder effects analogous to Griffiths phases and McCoy–Wu singularities. Among other topics also discussed here are criteria for crossovers, and non-equilibrium transitions and dynamics on diluted and possibly scale-invariant backgrounds.

As well as mappings, direct approaches of various sorts are used, such as mean field methods, and scaling procedures typically scaling a disorder distribution. Scaling for the non-equilibrium dynamics is relatively simple and very effective when the equations of motion are linear or linearizable by certain transformations presented here. The current situation regarding scaling approaches to the more resistant non-linear cases is briefly discussed.

1. Introduction

We know from familiar everyday examples, such as traffic, that even weak disorder can very strongly affect non-equilibrium transitions (such as jamming) and non-equilibrium collective dynamics. It is important to study such effects to find out what types of non-equilibrium phenomena occur in the presence of disorder and how they arise, and to provide quantitative descriptions. With a better understanding of the range and nature of the effects one can hope to elucidate mechanisms for a variety of unexplained empirical phenomena, for which disorder might be a possible source. The question whether it is an essential ingredient, in the sense of

being uniquely capable of producing observed effects, is an important issue for many topics (such as glassy behaviour).

This review attempts to give an overview of some of the fundamental issues, including some which are as yet only partially understood. So it is not intended to be a conclusive or comprehensive review but rather a survey of the current state of this rapidly developing area, with typically rather little detail. For some of the better understood parts details are or will soon be available in joint articles reporting collaborative investigations with Grynberg, Harris and Kaski.

The emphasis is on disorder effects in collective non-equilibrium systems, particularly in low dimensions. Many of the effects have parallels elsewhere, and particularly in disordered equilibrium spin models. This is because the basic models for strongly interacting non-equilibrium behaviour are lattice-based exclusion models [1, 2] which can be mapped to (quantum) spin models [1]. A wide variety of disorder effects have already been discussed for these latter models, and they help us to identify the corresponding phenomena in the non-equilibrium systems. The connections also often help to find ways to treat the non-equilibrium disorder effects, by taking over or generalizing techniques for the random spin systems or by devising their direct analogues. The emphasis will be on analytical approaches rather than computational ones.

The types of disorder phenomena which will be discussed (in the sections indicated) for the non-equilibrium collective systems include the following effects seen also in the disordered spin systems: localization (sections 3, 5–7), crossover to new critical behaviour and the criterion for such crossover (sections 3, 4, 7), criticality or crossover induced by scale-invariant geometry in diluted systems (sections 3, 4), Griffiths phases (section 7) and singular critical behaviour (of, for example, the McCoy–Wu type, see section 7).

The non-equilibrium stochastic models we discuss, i.e. the lattice-based exclusion models, have collective behaviour due to hard core interactions preventing double occupation of lattice sites, and these models are introduced in section 2. They have very interesting dynamical behaviour, particularly in the presence of disorder or dilution (sections 3–6), and also steady state phase transitions [1, 2] which are strongly modified by disorder (section 7).

In their mapping to quantum spin systems, spin up or down represents a particle or vacancy at a site, and the quantum spin Hamiltonian gives the evolution operator (section 2). The non-equilibrium steady state corresponds to the ground state of the quantum spin system, and 'quantum transitions' in the latter correspond to steady state non-equilibrium phase transitions. The strong interactions in such systems make them usually intrinsically non-linear, but there are linearizing transformations for special cases. We employ here two such linearizing transformations, which still apply in the presence of disorder. These are a transformation of a class of models to free fermion form [3] (section 5) and a Cole–Hopf transformation [4] on a mean field (mf) continuum model having a steady state phase transition [5] (sections 6 and 7). If the system can be mapped to linear form, the only remaining complication is disorder. For the linear systems this can often be treated by scaling techniques, particularly those scaling a distribution of random variables [6] (rates, in the systems considered here). Such techniques are illustrated in sections 4–6. MF approximations are employed in sections 6 and 7 and they are adequate in cases considered there, where strong disorder effects are of more concern than, for example, critical exponents.

In this overview only quenched substitutional disorder is considered. Annealed disorder is important in non-equilibrium systems with, e.g., freezing, but this is not discussed here. Frustration in its usual sense is not an issue here, as rates (the analogue of exchange interactions in the equivalent spin system) are always positive.

The first of the two main types of quenched disorder represented here is weak randomness. This can give rise to localization [7], Griffiths phases [8] and crossover to new critical behaviour including McCoy–Wu type singularities (sections 3, 5–7). The second type is dilution. This produces the scale-invariant geometry of the percolation network at the threshold concentration [6], which is responsible for critical slowing and crossover to new dynamic exponents (sections 3–5). In addition single defects are considered in sections 5 and 7.

2. Models and mappings

The lattice-based stochastic exclusion models [1, 2], employed throughout this review, allow each site l of a chosen lattice to be occupied by at most one particle. This condition effectively represents the action of a hard core repulsion which is responsible for the collective nature of the system. In addition a stochastic dynamic rule ('process'), observing the hard core 'exclusion' constraint, completes the definition of the model. The model is therefore described by a Master equation with appropriate transition probabilities (which need not satisfy detailed balance), or by an equivalent evolution operator (see below).

The simplest particle-conserving dynamic rule is nearest neighbour (nn) hopping where a particle can move to an empty site across any bond $l, l+1$ with probability per time step p_l, q_l for a move to the right or left, respectively. Despite its apparent simplicity, the pure version of this model has (even in dimension $d = 1$) a steady state phase transition [5] for $p_l \neq q_l$. The fully asymmetric case (ASEP), where hopping is only in one direction, or the symmetric case ($p_l = q_l$) are particular cases of great interest and are much emphasized in this overview. The pure ASEP, for example, maps exactly to a growth model whose continuum limit is the Kardar–Parisi–Zhang system [9], which has further well known wider connections. The processes just described can be combined with other non-conserving ones, such as single-particle or pair annihilation or production [10] ('evaporation' or 'deposition'). (Here and for the rest of this section we use one-dimensional cases for ease of presentation.)

So we are considering kinetic lattice gases with, in general, disordered (l-dependent) rates. Static lattice gases have been much considered as models of strong interaction phenomena, such as the liquid–gas transition. They are well known to map to Ising models through the introduction of a (pseudo) spin variable σ_l^z at each site l, related to the occupation number n_l ($= 1$ or 0 for particle or vacancy at site l) through the equation $\sigma_l = 2n_l - 1$. Such a mapping can be extremely useful also for the dynamic generalizations provided by the stochastic exclusion models, particularly for the case of disorder since so much is known about disordered spin systems. For the generalized mapping we need also a spin representation of the dynamic process. This is achieved [1] by writing the evolution operator in the form e^{-Ht} where the Hamiltonian H is that function of the spin variables which produces the dynamic process of interest. Typically each step of a given process removes particles from occupied sites and/or puts particles onto empty sites. Spin flip operators $\sigma_l^{\pm}$ produce these effects, so H contains combinations of such operators multiplied by appropriate rates (such as p_l, for bond $l, l+1$ in the case of the ASEP). H is therefore a quantum spin Hamiltonian in general. But it also must include σ_l^z operators through projection operators $P_l^{\pm} \equiv \frac{1}{2}(1 \mp \sigma_l^z)$ required to specify those configurations which might have changed, but did not (with probability $1 - p_l$ for the ASEP example) because of the stochastic nature of the process.

As a result, the quantum spin Hamiltonian representing the (bond-disordered) ASEP in $d = 1$ is [11]

$$H_{\mathrm{ASEP}} = H^{-+}\{p_l\} \equiv \sum_l H_{l,l+1}^{-+}(p_l) \equiv -\sum_l p_l[\sigma_l^- \sigma_{l+1}^+ - P_l^- P_{l+1}^+]. \qquad (2.1)$$

With an obvious generalization of the notation here introduced, the Hamiltonian for the partially asymmetric case is

$$H_{\mathrm{PASEP}} = H^{-+}\{p_l\} + H^{+-}\{q_l\}. \tag{2.2}$$

For the special case of symmetric hard core hopping, where $p_l = q_l$, the Hamiltonian reduces to that of the (bond-disordered) Heisenberg ferromagnet, and this will be important for results in sections 3 and 4. For the case of dimer deposition and evaporation (pair creation and annihilation) at rates ε_l, ε'_l for bond $l, l+1$, the equivalent quantum spin Hamiltonian is [10]

$$H_{\mathrm{DE}} = H^{++}\{\varepsilon_l\} + H^{--}\{\varepsilon'_l\}. \tag{2.3}$$

This also reduces, for the 'equal rate' special case $\varepsilon'_l = \varepsilon_l$, to the Heisenberg Hamiltonian, using a spin reversal transformation on one sublattice.

A generalized case of interest in what follows is the combination of the particle hopping and dimer deposition and evaporation processes, comprising $H_{\mathrm{PASEP}} + H_{\mathrm{DE}}$. The special case where $p_l + q_l = \varepsilon_l + \varepsilon'_l$ is reducible to free fermion form [3] (see section 5). The sub-case where $p_l = q_l = \varepsilon_l = \varepsilon'_l$ is the Ising model [12] (see section 4).

The above gives some specific examples of the general connection between stochastic non-equilibrium particle exclusion processes and quantum spin systems, disordered or not. There the role of the Master equation is taken over by the quantum spin Hamiltonian. This in general is not Hermitian but the real part of its eigenvalues is always non-negative. Also its ground state corresponds to the steady state of the non-equilibrium system, so steady state phase transitions (e.g. in the ASEP) correspond to 'quantum' ($T = 0$) transitions in the equivalent spin system.

Finally, we note that direct methods, not employing the mapping but instead proceeding from the Master equation or from the resulting hierarchy of equations for correlation functions, can often be very effective. Such is the case where the hierarchy of equations uncouple (cf the Heisenberg model), or where mf uncouplings simplify them. Examples of such direct approaches occur in several subsequent sections, particularly sections 6 and 7, where the corresponding disordered quantum spin model has not been previously studied or is poorly understood.

3. Symmetric hard core diffusion; disorder and dilution effects

Among the simplest collective non-equilibrium stochastic processes is symmetric hard core hopping, i.e. the symmetric exclusion process (SEP). As remarked above, the equivalent quantum spin system is the Heisenberg model. Disorder effects have been much studied in this spin model, and we can exploit and build upon these studies here.

Symmetric hard core diffusion does not have a steady state transition, so (unlike, e.g., section 6), the main interest here is in dynamic properties, particularly at long times when various critical effects can occur.

An important feature of the model is the Goldstone symmetry apparent in the equivalent Heisenberg system as the invariance under a common rotation of all spins. This symmetry has been exploited, first for the pure equal-rate dimer evaporation deposition process [10], then for the pure SEP [13] and more recently for the substitutionally disordered generalizations [14], to relate non-instantaneous correlation functions at finite particle densities to single-particle excitations in the Heisenberg ferromagnet. This remarkable simplification occurs because the symmetry allows a spin state with macroscopic magnetization (corresponding to finite density in the equivalent particle model) to be rotated into a state in the single spin wave sector. This simplifying relationship is used throughout this section.

Before getting into details we establish some notation while providing a few reminders about critical dynamics. These concern relationships between long characteristic times τ and lengths L, typically in the form

$$\tau = D^{-1}L^{z} \tag{3.1}$$

where the dynamic exponent z plays the role of the length scaling 'dimension' of time. L can be a probe length, e.g. a wavelength or some other long characteristic length (e.g. ξ as defined below). D is some rate 'constant'; it may change critically with disorder (see below). For the single-particle excitations of the pure Heisenberg ferromagnet (spin waves) $z = 2$, in any dimension d. This is consistent with the usual result for pure single-particle diffusion and also with the less well known fact that $z = 2$ holds also for hard core diffusion at finite particle densities (because of the result from the Goldstone symmetry argument).

The dynamic exponent value $z = 2$ remains for simple, e.g. mixed, bond randomness, leaving aside possible localization (see sections 5–7). A more interesting type of disorder, which we concentrate on for the rest of this section, is dilution, i.e. removal of bonds or sites at random [15]. This introduces, in addition to any probe length L, a further characteristic (correlation) length ξ which diverges like

$$\xi/a = c|x - x_{\mathrm{c}}|^{-\nu} \tag{3.2}$$

as the bond or site concentration x approaches a threshold ('critical') value x_{c}. The lattice constant a has been used to provide a reference length for later use. The divergence of ξ reflects the self-similarity of the geometry at the threshold.

For equilibrium dynamic processes in $d > 1$ the self-similar geometry induces critical behaviour in the dynamics of dilute systems near x_{c}, causing the softening of modes ($D = D(x) \propto \xi^{-\zeta} \to 0$ as $x \to x_{\mathrm{c}}$) and the eventual crossover at x_{c} to anomalous power-law dependences [6] characterized by modified dynamic critical exponents $\bar{z}$ where $\bar{z} = z + \zeta$.

In particular the dynamic scaling forms typically taken (at large times and lengths) by the characteristic time τ, and by the time- and space-dependent pair correlation function C are

$$\tau = L^{\bar{z}} f(L/\xi) \tag{3.3}$$

$$C(L, t, \xi) = L^{\cdots} g(t/\tau, L/\xi). \tag{3.4}$$

The universal function f in (3.3) crosses over between saturating and power law limiting forms: for example $f(x)$ tends to a constant as x tends to zero, so that a form like (3.1) is recovered at the percolation threshold ($\xi \to \infty$) but with modified exponent $\bar{z}$. The unspecified exponent in (3.4) involves the scaling dimension, at the threshold, of the dynamic variable occurring in the correlation function. In the case of spin waves, that variable is the transverse spin component. In that case D is the spin wave stiffness, which is related by an Einstein relation to percolation conductance and percolation density, which vanish like known powers (t and β) of $(x - x_{\mathrm{c}})$ as $x \to x_{\mathrm{c}}$. That determines the exponent ζ, making [6] (since $z = 2$)

$$\bar{z} = 2 + (t - \beta)/\nu. \tag{3.5}$$

However, we have the relation (for any disorder and any density) of the non-equilibrium collective system (hard core diffusion) to single spin wave properties. So hard core diffusion on diluted systems near the percolation threshold has the same scaling forms (3.3) and (3.4) with precisely the same scaling exponents and the same function f. Also the function g is the same in the two systems apart from an extra factor $\rho(1-\rho)$ appearing in the hard core diffusion case from the spin rotation to finite density [14], and an analytic continuation between real and imaginary 'times'.

Actually the function $C(L, t, \xi)$ had not been calculated for the spin wave system since what is of interest there is the dynamic structure factor $G(k, \omega, \xi)$, which is an inverse Fourier–Laplace transform of C (k and $\mathrm{i}\omega$ conjugate to L and t). An explicit form of G is known for

spin waves on dilute chains [16] and broad characteristics of its delicate structure are confirmed by experiment [17]. Obtaining the function C for hard core symmetric diffusion on diluted chains via transformation of G is very difficult, and it is easier to calculate C directly (and exactly) by considering the non-equilibrium collective dynamics on the statistical assembly of finite segments into which the chain breaks under dilution [14]. That results in the following dynamic scaling form (cf (3.4)) for the autocorrelation function at large t, ξ:

$$\overline{\langle n_l(0)n_l(t)\rangle} \equiv C(0,t,\xi) = bF(T)T^{1/6}\xi^{-1}. \tag{3.6}$$

Here the angular brackets denote average over histories, the bar denotes configurational average and

$$T = t/\xi^2 \qquad F(T) = \exp[-aT^{1/3}] \tag{3.7}$$

so $\bar{z} = z = 2$. In (3.7), a and b are known constants and ξ is the one-dimensional correlation length (which turns out to be related to the concentration variable x by $x = \mathrm{e}^{-1/\xi}$, so $x_c = 1$, $\nu = 1$). This stretched exponential form agrees well with simulations in the asymptotic regime. Of course, the underlying break up into distinct segments corresponds to a primitive (classical) form of localization of the dynamics.

So, we have begun by considering dilution, which is among the best understood of the specific disorder types, and has dramatic effects (dilution-induced criticality). The particular stochastic process (hard core diffusion) was chosen because of the exact equivalence to the simple linear spin wave problem. In any such case where the system dynamics can be reduced to linear equations, the only difficulty remaining concerns the disorder. Other reductions to linear form will be introduced in sections 5 and 6. So in the next section we turn to scaling techniques for dealing with the disorder, which are particularly simple for systems described by linear equations.

4. Scaling for critical and non-equilibrium dynamics

This section develops from a simple illustration of dynamic scaling in lattice systems with linear equations of motion [6]. We start with $d = 1$, and no disorder, where (by the relation to single spin excitations)

$$-su_l = p(u_{l-1} - u_l) - p(u_l - u_{l+1}) \tag{4.1}$$

covers the hard core diffusion process discussed in section 3. Here p is the hopping rate, s is the Laplace transform variable conjugate to time and u_l is a particle probability for site l. The philosophy applies also to other non-equilibrium stochastic processes with equations which are linear, or transformable to linear ones (as illustrated in examples in sections 5 and 6). The basic idea is to find and exploit the rescaling of dynamic variables, such as s, under a length rescaling achieved by dilating the lattice by removing ('decimating'), for example, alternate sites on the lattice. Generalizations to disorder [6] and to higher dimensions [6] will be briefly discussed below. For the case of dilution, contact can be made with the preceding section. There p becomes a random variable which also rescales, and this scaling of a static variable is sufficient to obtain percolation properties like the form of ξ, including the exponent ν. Scaling of static parameters, such as ratios of rates, are the way in principle to discuss steady state non-equilibrium transitions, and one [18] of the recently developed scaling methods for non-linear non-equilibrium systems (referred to at the end of section 7) proceeds in that way.

The 'decimation' (elimination) of u_l's with l odd (say) from the set of equations (4.1), provides an equation relating u_{2l} to $u_{2l\pm2}$ which is identical in form to (4.1) except for the change [6]

$$\Omega \equiv s/p \to \Omega' = R(\Omega) \equiv 4\Omega - \Omega^2. \tag{4.2}$$

At the (scale-invariant) fixed point, $\Omega^* = 0$, of (4.2) the small difference variable $\Omega - \Omega^*$ ($= \Omega$ in this case) rescales by the dynamic eigenvalue $\lambda = \mathrm{d}R(\Omega^*)/\mathrm{d}\Omega^*$ while the associated change in the lattice constant a (our reference length) is $a \to a' = ba$ with $b = 2$. So the length scaling exponent for Ω can be identified as z where $b^z = \lambda$. In the present case we thereby recover the proper z ($= 2$), which of course in this case is available by simpler direct methods.

In the generalization to substitutional bond disorder, the two p's in (4.1) become independent random variables since they relate to different bonds. Denoting their ratios to s by $r_{l-1,l}$, $r_{l,l+1}$ the decimation procedure provides scaling relations for renormalized variables $\{r'_{2l,2l+2}\}$. These become correlated but we can follow numerically the evolution of these random variables to obtain the dynamic critical behaviour of the disordered (linear) system [19]. Section 6 quotes an example of the results from this full procedure, actually applied to a non-linear non-equilibrium system made linear by a special transformation, and in which localization effects are discovered.

When correlations may be neglected a simpler approach, which allows more physical insight, is to rewrite the scaling of the random variables $\{r_{l,l+1}\}$ as a scaling of their distribution [6]. This evolves towards universal forms. The form for bond dilution (which is the case discussed in section 3 where dilution-induced critical effects occur) can be characterized by the weight x and the mean $1/\Omega$ of the non-zero values of $r_{l,l+1}$. Their joint scaling for dilatation by $b = 2$ is [6]

$$x' = x^2 \tag{4.3}$$

$$\Omega' = (1-x)^2(3\Omega - \Omega^2) + x(1-x)\Omega(8 - 9\Omega + 2\Omega^2)/(1-\Omega) + x^2(4\Omega - \Omega^2) \equiv \Delta(x, \Omega). \tag{4.4}$$

These scaling transformations have a (doubly unstable) fixed point at $x^* = 1$ ($= x_c$), $\Omega^* = 0$. By evaluating $\lambda \equiv \mathrm{d}\Delta(x^*, \Omega^*)/\mathrm{d}\Omega^*$ and the corresponding eigenvalue (2) for the linearized rescaling of x, one arrives at the scaling forms (3.2) and (3.3) with $\bar{z} = 2$, $\nu = 1$. But the full equation can, in principle, also give full functional forms. For example, since (4.3) results from a scaling which maintains the absolute correlation length ξ while $a \to a' = 2a$, it implies the full (exact) form $(a/\xi) \propto \ln(1/x)$. Thus one can, in principle, obtain the full dynamics of the system by scaling back from the dynamics of the equivalent single renormalized bond obtained by iterating the scaling equations sufficiently many times. In the most interesting (critical) case, where the (original) system has x near $x_c = 1$ and Ω small (corresponding to large ξ and large time t), under the forward iteration x moves gradually from near 1 towards the attractive fixed point $x^* = 0$ and, as a result, the map for Ω changes from the (chaotic) quadratic map $\Omega' \sim (4\Omega - \Omega^2)$ associated with almost extended state dynamics towards shorter and shorter cyclic behaviour corresponding to dynamics localized on smaller and smaller finite segments [20]. In this way the scaling equations exhibit (approximately because of the neglect of correlations) the characteristics of the exact 1D solution outlined in section 3.

For higher-dimensional (linear) systems the transformation of the equations of motion, by eliminating sites on a sublattice, normally leads to a proliferation of couplings. These may be suppressed by bond moving techniques [6], which combine contributions from the decimated equations of motion so as to provide a matching to a renormalized equation of the original form. The resulting rescaling of the random rates leads again to transformation equations for the associated distribution functions and, thereby, for their characteristic variables, e.g. the weight x and mean $1/\Omega$ defined above for the case of dilution, or a mean and width for the case of weak randomness (etc). The application of such procedures to symmetric hard core diffusion on, e.g., the bond-diluted honeycomb lattice follows procedures given

previously for dilute Heisenberg systems [21] and gives qualitative support for the scenario described in section 3. It gives, in particular, $z = 2.76$, in agreement with the value obtained by inserting into (3.5) exact and simulation results for the static percolation exponents ν, β and t (respectively) or by employing simulation results for the fractal and spectral dimensions [22].

All the above discussion is essentially a translation (due to the equivalences) of ones given earlier for Heisenberg magnets. The exact equivalence mentioned under (2.3) also allows the equal-rate dimer evaporation–deposition process to be discussed similarly. Corresponding applications to further stochastic systems differ in detail generally involving equations not previously studied in the context of disordered magnets (see sections 5 and 6). However, the same general procedure applies if the systems involve, or can be transformed to, linear dynamic equations. Two interesting non-linear cases which are transformable to linear equations (by Jordan–Wigner and Cole–Hopf transformations, respectively), and thereby amenable to scaling, are included in sections 5 and 6. For intrinsically non-linear problems the solution is much more complicated.

Some care has to be taken with such translations. Consider stochastic systems which, in some sense, map to the Ising magnet. One example (section 2) is symmetric hopping and dimer evaporation and deposition, all at the same rates($p = q = \varepsilon = \varepsilon'$) where the Ising Hamiltonian describes the evolution (section 2). Another is low-temperature Ising Glauber dynamics (of domain walls) which is equivalent to the same stochastic model but with rates, satisfying $p + q = \varepsilon + \varepsilon'$, determined by energetics of domain wall processes [23]. Scaling studies [24] and neutron scattering [25] on the dilute Ising model at low temperatures near the percolation threshold x_c show that, in place of $\tau \propto L^z$, the characteristic time to diffuse a thermal or percolation correlation length (ξ_T or ξ) is a scaling function of the two lengths which at x_c crosses over to $\tau \propto \xi_T^{(\ln \xi_T + \text{constant})}$ corresponding to a diverging dynamic exponent as $\xi_T \to \infty$. This arises from the self-similar branching of the percolation network at x_c. Here it is the Arrhenius relation of rates to energy barriers, together with the addition of barriers at a branching (giving a multiplication of small rates), which gives the anomalous slowing. It might be inferred that this extreme form of critical slowing would occur in both the stochastic models referred to above, with their correspondences to the Ising model. The inference is, of course, correct for the case of the (entirely equivalent) stochastic model with energetically determined rates. However, the inference is not correct for the (usual) stochastic model where the rates have no detailed balance association to energy barriers. In this usual (non-Arrhenius) stochastic 'analogue' the small rates are additive, so slowing of the standard form applies.

In general, disorder (in energy barriers) can have very strong effects on activated dynamics. In particular, it can be a source of anomalously slow or glassy dynamics, and of jamming.

5. Free fermion reductions

Under certain circumstances various quantum spin chains and one-dimensional stochastic Hamiltonians can be reduced to free fermion form by Jordan–Wigner transformations [26]. The best known examples are transverse Ising and XY models [27] and, among stochastic systems, the exclusion model with asymmetric hopping and pair annihilation and creation [3,12], which was introduced in section 2. These free fermion reductions, all for the pure case, also generalize to certain sorts of disorder, and that is the topic of this section.

Section 2 gives the Hamiltonian $H = H_{\text{PASEP}} + H_{\text{DE}}$ for the bond-disordered generalization of the stochastic model just mentioned. Here the rate variables for the lth bond are $p_l, q_l, \varepsilon_l, \varepsilon'_l$.

Under the Jordan–Wigner transformation of spin variables for site l to fermion creation and annihilation operators c_l^+, c_l the Hamiltonian turns into a quadratic part (from $\sigma^\pm\sigma^\pm$ terms and σ^z terms) together with a quartic part (from $\sigma^z\sigma^z$ terms). The quartic terms disappear if

$$p_l + q_l = \varepsilon_l + \varepsilon'_l. \tag{5.1}$$

This is the condition for a free fermion result in the general bond-disordered case. A subcase is the bond-disordered Ising Hamiltonian resulting when $p_l = q_l = \varepsilon_l = \varepsilon'_l$. For the more general case where (5.1) applies the resulting quadratic fermion Hamiltonian is

$$H = \sum_l \{p_l c_l c_{l+1}^+ - q_l c_l^+ c_{l+1} - \varepsilon_l (c_l^+ c_{l+1}^+ - 1) + \varepsilon'_l c_l c_{l+1} + [p_{l-1} - p_l + \varepsilon_l - \varepsilon'_{l-1}] c_l^+ c_l\}. \tag{5.2}$$

The asymmetry between p_l and q_l in the last coefficient is due to having used (5.1) to give a short form.

In the pure case where the rates are l-independent, Fourier transformation followed by a Bogoliubov–Valatin transformation takes this to diagonal free fermion form. The resulting spectrum is [3]

$$\epsilon_k = \varepsilon + \varepsilon' + (\varepsilon - \varepsilon') \cos k + \mathrm{i}(p - q) \sin k. \tag{5.3}$$

The spectrum has a gap, causing exponential decay, unless ε or ε' vanishes. Then it is gapless and includes diffusive and also ballistic behaviour (from real and imaginary parts of ϵ_k, respectively).

By such means the pure stochastic system has been solved for dynamical correlation functions [3], shock profiles [28] and for persistence probability, etc. Such a procedure also, of course, solves the corresponding [23] pure Ising model with single spin flip Glauber dynamics.

It appears from such studies, and from a mf argument, that the free fermion model discussed above does not have a steady state transition for ε and/or ε' non=zero, even with boundary driving (e.g. particle injection/ejection at the two ends), unlike the ASEP (see section 7). So (as in preceding sections) our discussion here is confined to the dynamic behaviour.

The simplest generalization with any disorder is a single defect in the free fermion stochastic system. This can be treated by standard techniques and the defect can give rise to the stochastic analogue of localized modes, etc [29].

The next simplest situation is dilution. This case can be treated by exploiting the decomposition of the chain dynamics into dynamics on finite segments (cf section 3). The effects of dilution are most interesting for the case where ε or ε' vanishes. This is the situation which is gapless in the pure version. On a finite segment of length L (occurring with probability proportional to $\exp[-L/\xi]$, with ξ the percolation correlation length) the spectrum has a finite-size-induced gap whose value is roughly Re ϵ_k at $k \sim 1/L$. Asymptotic analysis on the averaged autocorrelation function shows that, for large ξ and long times, this gap leads to a stretched exponential decay. This is very like that seen for symmetric hard core hopping on diluted chains (section 3) but it is now multiplied by a factor periodic in $t^{2/3}$, arising from the ballistic character of the motion [30].

In the single defect and 1D dilution examples just discussed Fourier transformation, followed by the Bogoliubov transformation, could be exploited since in both cases sections of the system remain uniform. For more general sorts of disorder Fourier transformation is not appropriate. Nevertheless a generalized type of Bogoliubov–Valatin transformation can be carried out directly on forms like (5.2) [31] as a preliminary to numerical calculations. Alternatively [32] we can construct the required transformation directly from the coupled linear dynamic equations in real space obtained by taking the commutators of H with c_l and c_l^+. With or without the transformation, scaling techniques of the type discussed in section 4

(but generalized to the case of coupled equations) can be applied directly to these disordered equations because they are linear.

A slightly different version of the scaling technique [19] will be briefly outlined in section 6 (for the uncoupled case), which is particularly effective. It shows very clear localization effects in the dynamics of the free fermion model with weak disorder [33]. As might be expected parallels exist between the behaviour of such disordered free fermion stochastic systems and phenomena seen in disordered free fermion quantum spin systems [34]. Such analogies are particularly interesting for stochastic systems with steady state transitions (see section 7).

6. Asymmetric exclusion process, Cole–Hopf transformation and localization

This section is concerned with effects of disorder on the ASEP. This model is representative of non-linear stochastic systems, and it is the simplest non-equilibrium model having a steady state phase transition.

There are reasons to expect that disorder effects might be strong in the non-linear systems, and particularly near any steady state transitions. This issue will be discussed in greater detail in section 7. In the present section we limit the discussion to disorder effects on the dynamics, away from such transitions, after presenting a simplified approach to the model.

The ASEP has been defined in section 2. It is the $\varepsilon = \varepsilon' = 0$ special case of the model discussed in section 5, and does not satisfy the condition for reduction to free fermion form. Nevertheless its non-disordered mf continuum version can be linearized by the Cole–Hopf non-linear transformation [4] given below. A generalization of that transformation also applies for special sorts of disorder, and this will be exploited in the following discussion.

Many interesting features of the ASEP (including the transition) already occur in $d = 1$, and we discuss that case here. Because of the hopping process of the model, already specified in section 2, the discrete continuity equation giving the rate of change of the (average over histories of the) density $\rho_l = \langle n_l \rangle$ at site l is

$$\partial \rho_l / \partial t = J_{l-1,l} - J_{l,l+1} \tag{6.1}$$

where

$$J_{l,l+1} = \langle \{ p_l n_l (1 - n_{l+1}) - q_l (1 - n_l) n_{l+1} \} \rangle \tag{6.2}$$

is the current across bond $l, l+1$.

So far, things are exact. But, even for the pure case, exact reduction is limited to the steady state situation (see section 7). So we proceed further here (for the disordered dynamics) by employing a mf factorization of the RHS of (6.2), under which the current $J_{l,l+1}$ becomes

$$\bar{J}_{l,l+1} = p_l \rho_l (1 - \rho_{l+1}) - q_l (1 - \rho_l) \rho_{l+1}. \tag{6.3}$$

In the steady state situation $\partial \rho_l / \partial t = 0$, the current $J_{l,l+1}$ becomes a constant J (independent of l, t) and the mf version of the resulting steady state equation is [35]

$$\bar{J}_{l,l+1} = J. \tag{6.4}$$

This is equivalent to a first-order map for the profile ρ_l. The pure mf steady state transition, at a critical value J_c of J, is readily obtained from the pure version of (6.4) (section 7).

We proceed here in an alternative way, keeping the full dynamics but again using the mf reduction (6.3) and, in addition, a continuum approximation $\rho_l(t) \to \rho(x, t)$. In this mf continuum limit (6.1) becomes

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left[D(x) \frac{\partial \rho}{\partial x} - \lambda(x) \rho (1 - \rho) \right] \tag{6.5}$$

where $D(x)$, $\lambda(x)$ are continuum forms of $p_l + q_l$, $p_l - q_l$ respectively. Equation (6.5) is a disordered generalization of the noiseless Burgers equation [36].

It is well known that in the pure case (D, λ constant) the Cole–Hopf transformation

$$\rho - \frac{1}{2} = \frac{D}{\lambda}\frac{\partial \ln u}{\partial x} \tag{6.6}$$

linearizes (6.5), taking it to a diffusion equation for $u(x, t)$. That allows a full mf solution for the pure dynamical problem, which includes the pure steady state transition given by the profile map as a special case. For the pure system an exact solution for the steady state is available by other (operator algebra) means [37]. This confirms that a steady state transition does occur, and that the mf description based on (6.4) gives the exact phase boundaries (though not the correct exponents). Further, the mf continuum description of the dynamics yields moving kinks and other phenomena in qualitative agreement with exact results. So mf theory provides a reasonable framework in which to investigate the effects of disorder on the phase transition (section 7) and on dynamics (this section).

The simplest case of disorder in the 1D ASEP is a single defect. The continuum field description of this case is simple, using matching of Cole–Hopf solutions (a corresponding simulation result is quoted in section 7 for the steady state case). The next simplest is dilution, since as before this can be treated by superposing solutions (in this case obtained via the Cole–Hopf procedure) on finite segments [38]. For other cases we can proceed as follows.

For the fully asymmetric case ($q_l = 0$) with bond-disordered rates p_l, a generalization of the Cole–Hopf transformation (6.6) has been found [38] which takes the corresponding continuum equation ((6.5) with $\lambda(x) \propto D(x)$) to

$$\partial u/\partial t = D(x)\partial^2 u/\partial x^2. \tag{6.7}$$

This form being linear, scaling procedures in the same spirit as those in section 3 can now be applied to deal with the disorder. Because the right-hand side of (6.7) is not the more usual form $\partial/\partial x(D(x)\partial u/\partial x)$ the discrete version of (6.7) appropriate for scaling is not (4.1). Instead it is contained in an obvious generalization of the equations of motion used in a scaling treatment of dynamics in a one-dimensional Mattis-transformed Edwards–Anderson Heisenberg spin glass [19]:

$$u_l = g_l(V_{l,l-1}u_{l-1} + V_{l,l+1}u_{l+1}) \tag{6.8}$$

where $g_l \equiv (E_l - \varsigma_l s)^{-1}$. Here ς_l is the origin of the randomness (cf $D(x)$) and at the outset E_l and $V_{l,l\pm1}$ are non-random (l-independent). Under exact decimation scaling by $b = 2$ they transform according to

$$V'_{l,l+2} = g_{l+1}V_{l,l+1}V_{l+1,l+2} \tag{6.9}$$

$$E'_l = E_l - g_{l-1}V^2_{l,l-1} - g_{l+1}V^2_{l,l+1} \tag{6.10}$$

so they pick up correlated randomness. Under the numerical interaction of these scaling equations for the case of weak bond randomness the V's evolve chaotically or cyclicly inside a well defined exponentially decreasing envelope, which is an indication that the randomness causes localization of states. Under the inverse Cole–Hopf transformation back to the original system, the main consequence of the localized form of the state u is a shift of the profile ρ_l. Whether this is related to observations in low-dimensional deterministic non-linear systems of the relief of localization by non-linearity [39] is not presently understood.

Direct attacks on disorder effects in the dynamics of the ASEP, and related systems, can also be found in the work of Krug and others [40]. A complementary approach is provided by field theoretic techniques [41]. They have been used to study effects of weak randomness on dynamic exponents in noisy higher-dimensional generalizations of systems like (6.5). Such approaches

keep the fluctuations which are suppressed in mf theory and so are capable of providing exact results for universal quantities.

7. Steady state transitions, Griffiths phases, etc

This section discusses some effects of disorder on steady state transitions, again taking the asymmetric exclusion model as the representative example.

It can be argued that disorder should be expected to have extreme effects on the steady state transitions of stochastic non-equilibrium exclusion models, and on the collective systems they model. This is because of the mapping of their non-equilibrium steady states to the ground states of quantum spin systems [1,2]. That implies that the steady state transitions correspond to 'quantum' (zero-temperature) transitions in quantum spin models. These latter models can, in turn, be mapped to higher-dimensional classical spin models [27] (by well known methods employing the Trotter formula or taking Hamiltonian limits, etc). Randomness in the original non-equilibrium model then corresponds to striped randomness in this higher-dimensional classical model which (following the results of McCoy and Wu [42] and intuitive considerations) would be expected to produce highly singular critical behaviour. Additional independent evidence for extreme effects of disorder on quantum phase transitions comes from scaling treatments of models such as the transverse Ising chain [43].

Also, near any transition (quantum or not) disorder should be expected to cause Griffiths singularities [8]. An example indicating how such singularities arise is a disordered (classical) spin system with, for example, bond-disordered exchange interactions. It can contain, with low probability, large regions of weak/strong bonds, in which the (local) ordering and associated weak singularities occur at temperatures below/above the bulk transition temperature. The corresponding phenomena in non-equilibrium systems with steady state transitions is well illustrated with the ASEP [44]. There the argument given above applies, except that now the current J plays the role of temperature (see (6.4) and immediately below) and the rates p_l are analogous to the exchange interactions in the spin system example. The ordered and disordered regions of the spin system correspond to local high current and low current phases in the ASEP. These phases have a further interpretation within the description of equations (6.5)–(6.7) using the Cole–Hopf transformation. The steady state high current and low current phases are distinguished there by whether the operative solutions for $u(x, t)$ are extended or localized. In the steady state ASEP, as usual, the Griffiths phases correspond to unlikely inclusions of the minority phase in the majority one.

This feature can also be easily seen in the following discussion [44] of the steady state mf ASEP, with disorder, proceeding directly from the discrete non-linear profile map equivalent to (6.4) (rather than via the linearizing Cole–Hopf transformation). We take the fully biased case ($q_l = 0$) where (6.4) and (6.3) give

$$\rho_{l+1} = 1 - J/(p_l \rho_l) \equiv M_{l,J}(\rho_l). \tag{7.1}$$

Iteration of this map gives the l-dependent density ρ_l. The result is uniquely determined by J together with one boundary condition; or by two boundary conditions (e.g. open boundary conditions involving injection and ejection with rates α, β at left and right boundaries, respectively) which also determine J.

For the pure case where the mapping function M is independent of l, the map has two, one or no fixed points for J less than, equal to, or greater than the critical current $J_c = p/4$ at which the steady state transition occurs. In the low current phase (ignoring possible proximity of boundaries) the profile ρ is a kink in which ρ approaches constant values ρ_L or ρ_R far to the left or right of a region, of width $\propto (J_c - J)^{\frac{1}{2}}$, where ρ_l increases from about ρ_L to ρ_R.

This corresponds to real exponentials in the Cole–Hopf function u. In the high current phase ρ_l decreases with l and is close to $\frac{1}{2}$ except near boundaries (corresponding to imaginary exponentials in u). In both phases the profile has characteristic lengths diverging like $|J - J_c|^{\frac{1}{2}}$ as $J \to J_c$. In the disordered case M depends on l (through p_l) so the mapping differs from iteration to iteration. J_c is now an (unknown) function of all the p's. At any J, e.g. near J_c for a given bond configuration, the iteration can, with low probablility, involve a long succession of (J/p_l)'s, each less than $\frac{1}{4}$ and elsewhere a succession of values all greater than $\frac{1}{4}$, corresponding to the Griffiths inclusions. These will show, respectively, an increasing or decreasing trend for the profiles ρ_l, characteristic of the two different phases. This actually implies that J_c itself is largely determined by the Griffiths inclusions with small p_l.

Results from numerical iterations of (7.1) for the disordered one-dimensional steady state ASEP show all these features, as do simulations of the full system (without the mf approximation). For a given configuration of random p_l's, the features in the profile correspond quite closely, suggesting that the mf description is adequate for such phenomena.

Simulation results have also been obtained for the steady state profiles of 1D ASEPs containing single defects. As an example, for a single weak bond in the centre of the system the boundary conditions can be such that across the weak bond the system behaves as if it were in the high current phase, while everywhere else it has kink forms (displaced with respect to each other on either side of the defect) characteristic of the low current phase. MF theory provides qualitatively the same features.

Now, for the pure 1D ASEP, the exact steady state properties are available from the operator algebra solution. In principle this allows some limited exact progress with disorder effects, particularly for weak disorder (in the steady state 1D ASEP).

In particular the shift of the pure J_c $(= p/4)$ under the effects of a low concentration of defects (e.g. rates $p_l \neq p$) is in principle calculable in terms of pure system correlation functions following procedures [45] introduced for spin systems. Also, again following arguments given first for spin systems [45], one can determine from pure properties whether the pure critical behaviour is unstable with respect to effects of weak randomness. For ordinary equilibrium transitions in general it is known that the 'Harris' criterion for this instability (implying in general a crossover to new random critical behaviour) is $\alpha > 0$ where $\alpha = (2 - d\nu)$ is the pure specific heat exponent. A corresponding criterion can be given for the relevance of weak disorder in non-equilibrium steady state phase transitions. We illustrate this with models (like the ASEP) where the exact solution for the pure model exhibits a transition at a critical current J_c, proportional to some rate p, near which a correlation length diverges like

$$\xi = |(J_c - J)/J_c|^{-\nu}. \tag{7.2}$$

Central limit considerations for a correlation volume suggest that in d dimensions introduction of uncorrelated weak disorder into the rate p causes a shift of J_c by $\Delta J_c \sim J_c \xi^{-d/2}$. For

$$2 - d\nu > 0 \tag{7.3}$$

this shift clearly swamps the pure difference $|J - J_c| = J_c \xi^{-1/\nu}$ associated with the finite correlation length. So (7.3) is the criterion for randomness to modify the critical behaviour at the non-equilibrium steady state transition. As an example, in the 1D ASEP, where $\nu = 1$, the steady state critical behaviour is modified by weak disorder.

In such a case, scaling methods would be the obvious way of finding the new steady state critical behaviour. In analogous cases, like the quantum magnets where disorder effects can be severe, direct real space scaling methods have proved effective [43], so it would be helpful to use them for the non-equilibrium transitions. Two approaches are presently being developed, the first of which is a generalization to the disordered case of a procedure which

scales the equivalent quantum system [46]. The other is a disordered generalization of a recently developed direct scaling method for non-linear non-equilibrium systems [18], applicable to both steady state and dynamic behaviour. This method keeps very close contact with the physics of the stochastic process, allowing easier interpretation and fuller intuitive understanding.

8. Conclusion

This review of disorder effects in non-equilibrium systems has been confined to substitutional disorder in lattice-based stochastic exclusion processes. Clearly these are highly oversimplified models. Nevertheless they are collective and without detailed balance and already show a remarkable range of properties in their pure behaviour. Intuitive ideas suggested that they could be very strongly affected by disorder, and some such effects have already been seen. So the disordered versions are potentially a very rich class of systems. However, the coverage in this review has been very limited, and that reflects our present state of understanding of models with quenched spatially located disorder. (The simpler, but still very interesting, case of particle-associated randomness is better understood [47].)

We have discussed only the simple members of the classes which have so far been studied. Nevertheless even those show a wide range of phenomena. Many such phenomena were suggested or established by the equivalence to well studied disordered spin models. This helpful correspondence has taught us much, so that has been one of the themes here. We obtain from it qualitative appreciation that such phenomena as localization, new critical effects and new singular behaviours, Griffiths phases, etc, should be expected and also it provides one approach for quantitative descriptions.

Out of the possible exclusion models, we have discussed only disordered versions of single species models with two-site processes, and most of the discussions (all of them in the cases of section 5 and 6) have been for $d = 1$. All these are ripe for generalization, and many new phenomena can be expected. For example, the higher-dimensional models are more likely to show severe effects like jamming under the influence of disorder, and that is particularly expected in models with caging effects or multisite processes representing various kinetic constraints.

Also, as regards substitutional disorder, only the simplest types of (spatial) disorder have here been considered. Correlated (striped) randomness is now being investigated: as mentioned briefly, it occurs naturally in discussions of steady state transitions in non-equilibrium systems with uncorrelated randomness using their mapping to higher-dimensional classical spin systems. Temporal variations of rates play a role in some real situations.

There are clearly many prospects for generalizations of the models, but they will typically be harder to tackle analytically. Improved treatments or techniques are required both for these generalized models and for the more difficult questions with the simpler models (like descriptions of spatio-temporal variations in disordered systems with, for example, Griffiths phases). The required improvements could be merely generalizations of existing quantum spin techniques for previously unconsidered disordered models (see, e.g., the end of section 7), or development of new direct approaches for the disordered non-linear exclusion models, e.g. along the lines of the real space renormalization methods for scaling distributions (also referred to in section 7).

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